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Critical Properties of the Heisenberg Ferromagnet with Higher Neighbor Interactions ($S=\frac{1}{2}$)

N. W. Dalton
Wheatstone Physics Laboratory, King's College, London, England

AND

D. W. Wood
Department of Mathematics, University of Nottingham, Nottingham, England

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The method of exact power-series expansions has been extended to include both nearest-neighbor and next-nearest-neighbor interactions in the Heisenberg model. The series expansions for the susceptibility in zero magnetic field and the free energy in zero magnetic field have been derived to the fifth power in reciprocal temperature for the simple cubic, body-centered cubic, and face-centered cubic lattices. For the special case when all interactions are equal (equivalent-neighbor model), an additional term has been obtained in these expansions. For purposes of discussing the susceptibility and magnetic specific heat, the series expansions have been derived for lattices in which third-neighbor interactions are included, but only for the equivalent-neighbor model. Estimates of critical points are given, and the Padé-approximant method is used to study the dependence of the critical properties (temperature, energy, and entropy) on the relative strength of the first- and second-neighbor interactions. It is found that the variation in the critical point is well represented by

$$T_c(\alpha)=T_c(0)[1+m_1\alpha],$$

where $\alpha=J_2/J_1$ and lies in the range $0\leqslant\alpha\leqslant 1$, and $T_c(0)$ is the critical temperature of the nearest-neighbor model. The values of m_1 are 0.76, 0.99, and 2.74 for the fcc, bcc, and sc lattices respectively. Both the second-neighbor model and the equivalent-neighbor model are used to investigate the behavior of χ_0 for values of T near T_c. It is found that all the coefficients in the magnetic-specific-heat series expansion are positive for the equivalent-neighbor model, and that for lattices with large coordination numbers, reliable estimates of the critical point may be obtained using this function.

I. INTRODUCTION

Much previous work has been done on the critical behavior of the Heisenberg model of a ferromagnet when it is assumed that exchange interactions $(-2J\mathbf{S}_i\cdot\mathbf{S}_j)$ exist only between nearest-neighbor spins on the lattice. The most powerful theoretical approach towards obtaining estimates of critical constants is that introduced by Kramers and Opechowski.[1] In this method exact series expansions in ascending powers of reciprocal temperature are derived for the partition function and related thermodynamic functions for various lattice structures. In recent years much work has been done in extending the series expansions for the zero field susceptibility χ_0 and magnetic specific heat C_v to a high degree of approximation.[2] For the case where the spin variable S may take an arbitrary value the most extensive calculations have been performed by Rushbrooke and Wood.[3] These authors obtained the first six coefficients in the susceptibility series, and the first five coefficients in the magnetic specific-heat series. Recently a more powerful method of deriving these coefficients has been developed by

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